2011 offline

If $\omega (\neq 1)$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$ then (A, B)

(a) (1, -1) (b) (0, 1) (c) (1, 1)
D (1, 0)

$$(1+\omega)^7 = A + B\omega$$

$$(1+\omega)^7 = (-\omega^2)^7 = -\omega^{14}$$

$$= -(\omega^2 \cdot \omega^{12})$$

$$= -(\omega^2 (\omega^3)^4)$$

$$= -(\omega^2 \cdot (1)^4)$$

$$= -\omega^2$$

$$= 1+\omega$$

$$\Rightarrow 1+\omega = A + B\omega$$

$A = 1$ $B = 1$

(1, 1)

c

Let α, β be real and z is a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line $Re(z) = 1$, Then it is necessary that

(a) $\beta \in (1, \infty)$ (b) $\beta \in (0, 1)$

(c) $\beta \in (-1, 0)$ (d) $|\beta| = 1$

z is a complex number

$z^2 + \alpha z + \beta = 0$ has two distinct roots

and $Re(z) = 1$

$\Rightarrow (1 + ip), (1 - ip)$ are the roots of

$z^2 + \alpha z + \beta = 0$

product of roots $= \frac{c}{a}$

$$(1 + ip)(1 - ip) = \beta$$

$$1 + p^2 = \beta$$

$$1 + p^2 > 1$$

$$\Rightarrow \beta > 1$$

$$\beta \in (1, \infty)$$

(a)

2011

Q. $\omega \neq 1$ is cube root of unit and $H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ then H^{70} equals to

(a) 0 (b) $-H$ (c) H^2 (d) H

$$H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$$

$$= \omega \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \omega I$$

$$H^{70} = (\omega I)^{70}$$

$$= \omega^{70}$$

$$= \cancel{(\omega^3)^{23}}$$

$$= (\omega^3)^{23}\, \omega$$

$$= \omega$$

$$= \omega I$$

$$= H$$

(d)

DATE

2012 off line

Let z be a complex number such that the imaginary part of z is non zero and $a = z^2 + z + 1$ is real. Then a cannot take the value

(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

$$a = z^2 + z + 1$$

$$0 = z^2 + z + 1 - a$$

$$z = \frac{-1 \pm \sqrt{1 - 4(1-a)}}{2}$$

$$= \frac{-1 \pm \sqrt{1 - 4 + 4a}}{2}$$

$$= \frac{-1 \pm \sqrt{4a-3}}{2}$$

z do not have real roots because imaginary part is non zero

$$4a - 3 < 0$$

$$a < \frac{3}{4}$$

So a cannot take value $\frac{3}{4}$ (d)

2013 of-f

If z is a complex number of unit modulus and argument θ then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals

(a) $\frac{\pi}{2}-\theta$ (b) θ (c) $\pi-\theta$ (d) $-\theta$

$$|z| = 1 \text{ so } |z|^2 = 1$$

$$\Rightarrow z\bar{z} = 1$$

$$\bar{z} = \frac{1}{z}$$

$$\arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right)$$

$$= \arg\left(\frac{1+z}{\frac{z+1}{z}}\right)$$

$$= \arg\left(\cancel{(1+z)} \times \frac{z}{\cancel{(z+1)}}\right)$$

$$= \arg z$$

$$= \theta \text{ (given)}$$

(b)

2013 online.

Q. $z_1 \neq 0$ and z_2 are two complex numbers such that $\frac{z_2}{z_1}$ is purely imaginary number, then

$$\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right| \text{ is equal to}$$

(a) 2 (b) 5 (c) 3 (d) 1

$$\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|$$

divide by $3z_1$

$$\left|\frac{\frac{2}{3}+\frac{z_2}{z_1}}{\frac{2}{3}-\frac{z_2}{z_1}}\right|$$

$$\left|\frac{\frac{2}{3}+ip}{\frac{2}{3}-ip}\right| \quad \left[\begin{array}{l}\text{Given } \frac{z_2}{z_1} \text{ is imaginary} \\ \text{so } \frac{z_2}{z_1} = ip\end{array}\right]$$

$$1 \quad \left[\begin{array}{l}\because |z| = |\bar{z}| \\ \left|\frac{z}{\bar{z}}\right| = 1\end{array}\right]$$

DATE

2013 online

Q. a complex number z satisfies $z + \sqrt{2}\,|z+1| + i = 0$ Then $|z|$ is equal to

(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) 1

Q. complex number z satisfies $z + \sqrt{2}\,|z+1| + i = 0$

Then $\operatorname{im}(z) + 1 = 0$

$\operatorname{im}(z) = -1$

Let $z = a - i$ $\left[\begin{aligned} z &= a + ib \\ &= a + i(-1) \\ &= a - i \end{aligned} \right.$

$z + \sqrt{2}\,|z+1| + i = 0$

$(a-i) + \sqrt{2}\,|(a-i+1)| + i = 0$

$a - \cancel{i} + \sqrt{2}\,|(a+1) - i| + \cancel{i} = 0$

$a + \sqrt{2}\sqrt{(a+1)^2 + (-1)^2} = 0$

$a + \sqrt{2}\sqrt{a^2 + 1 + 2a + 1} = 0$

$\sqrt{2}\sqrt{a^2 + 2a + 2} = -a$

$2(a^2 + 2a + 2) = a^2$

$2a^2 + 4a + 4 = a^2$

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$$a^2 + 4a + 4 = 0$$

$$(a+2)^2 = 0$$

$$a + 2 = 0$$

$$a = -2$$

$$z = a - i$$

$$= -2 - i$$

$$|z| = \sqrt{(-2)^2 + (-1)^2}$$

$$= \sqrt{4+1}$$

$$= \sqrt{5}$$

2014 online

For all complex numbers z of the form $1+i\alpha$, $\alpha \in R$ if $z^2 = x+iy$ the

(a) $y^2-4x+2=0$ (b) $y^2+4x-4=0$

(c) $y^2-4x+4=0$ (d) $y^2+4x+2=0$

$$z = 1+i\alpha$$

$$z^2 = x+iy$$

$$(1+i\alpha)^2 = x+iy$$

$$1-\alpha^2+2i\alpha = x+iy$$

$$1-\alpha^2 = x \qquad 2\alpha = y$$

$$\alpha = \frac{y}{2}$$

$$1-\left(\frac{y}{2}\right)^2 = x$$

$$1-\frac{y^2}{4} = x$$

$$4-y^2 = 4x$$

$$y^2+4x-4=0$$

2014 Mains.

If z is a complex number such that $|z| \geq 2$ then the minimum value of $|z + \frac{1}{2}|$

(a) is equal to $\frac{5}{2}$

(b) lies in the interval $(1, 2)$

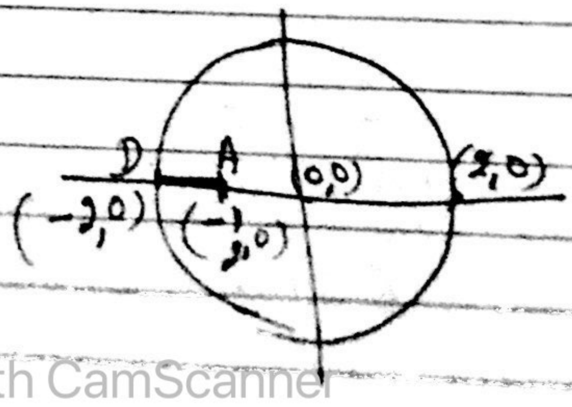
(c) strictly greater than $\frac{5}{2}$

(d) strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$.

$|z| \geq 2$

$\Rightarrow$ According to this region is on or outside circle whose centre is $(0,0)$ and radius is 2

Minimum of value of $|z + \frac{1}{2}|$ means distance AD.



$A\left(-\frac{1}{2}, 0\right) \quad D(-2, 0)$

$$AD = \sqrt{\left(-2+\frac{1}{2}\right)^2 + (0-0)^2}$$

$$\sqrt{\left(\frac{-3}{2}\right)^2} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$\frac{3}{2} \in (1, 2)$$

b

2015 off line

A complex number is said to be unimodular if $|z|=1$ Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular Then z_1 lies on a

(a) straight line parallel to x-axis
(b) straight line parallel to y-axis
(c) circle of radius 2
(d) circle of radius $\sqrt{2}$

$$\left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right| = 1$$

$$|z_1-2z_2| = |2-z_1\bar{z}_2|$$

$$|z_1-2z_2|^2 = |2-z_1\bar{z}_2|^2$$

$$|z_1|^2+4|z_2|^2-\cancel{2\bar{z}_1 z_2}-\cancel{2z_1\bar{z}_2} =$$

$$4+|z_1|^2|\bar{z}_2|^2-\cancel{2\bar{z}_1 z_2}-\cancel{2\bar{z}_1 z_2}$$

$$0 = -|z_1|^2-4|z_2|^2+4+|z_1|^2|\bar{z}_2|^2$$

$$0 = |z_1|^2|\bar{z}_2|^2-|z_1|^2+4-4|z_2|^2$$

$$= |z_1|^2(|\bar{z}_2|^2-1)+4(1-|z_2|^2)$$

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$$|z_1|^2(|\bar{z}_2|^2-1)-4(|z_2|^2-1)$$

DATE

$$\left(|z_1|^2 - 4\right)\left(|z_2|^2 - 1\right) = 0$$

$$|z_1|^2 = 4 = 0 \qquad |z_2|^2 - 1 = 0$$

$$|z_1|^2 = 4 \qquad |z_2|^2 = 1$$

$$|z_1| = 2 \qquad |z_2| = 1$$

But $|z_2| \neq 1$

$$\Rightarrow \quad |z_1| = 2.$$

z_1 lies on a circle of radius 2.

2016 online

Let $z = 1 + ai$ be a complex number
$a > 0$ such that z^3 is a real num
Then the sum
$1 + z + z^2 \ldots z^{11}$ is equal to

(a) $1365\sqrt{3}\, i$ (b) $-1365\sqrt{3}\, i$

(c) $-1250\sqrt{3}\, i$ (d) $1250\sqrt{3}\, i$

$$z = 1 + ai$$
$$z^3 = (1 + ai)^3$$
$$= 1 + 3ai - 3a^2 - ia^3$$
$$= 1 - 3a^2 + (3a - a^3)i$$

z^3 is a real number

$$\Rightarrow 3a - a^3 = 0$$
$$a(3 - a^2) = 0$$
$$a(\sqrt{3} - a)(\sqrt{3} + a) = 0$$
$$\Rightarrow a = \sqrt{3}$$

$$z = 1 + ai$$
$$= 1 + \sqrt{3}\, i$$
$$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$S = 1 + z + z^2 \cdot \cdot \cdot z^{11}$

$= \frac{1 - z^{12}}{1 - z}$

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z^{12} = 2^{12}\left(\cos 4\pi + i\sin 4\pi\right)$

$= 2^{12}(1 + i0)$

$= 2^{12}$

$1 - z = 1 - (1 + \sqrt{3}i)$

$= \cancel{1} - \cancel{1} - \sqrt{3}i$

$= -\sqrt{3}i$

$\frac{1 - z^{12}}{1 - z} = \frac{1 - 2^{12}}{-\sqrt{3}i}$

$= \frac{1 - 4096}{-\sqrt{3}i}$

$= \cancel{-}\frac{4095}{\cancel{-}\sqrt{3}i} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\overset{1365}{\cancel{4095}}\sqrt{3}}{\cancel{3}}i$

$= 1365\sqrt{3}i$

DATE

2016 offline

A value of θ for which $z = \dfrac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$ (c) $\sin^{-1}\dfrac{\sqrt{3}}{4}$ (d) $\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$$Re(z) = 0$$

i.e $z + \bar{z} = 0$

$$\frac{2+3i\sin\theta}{1-2i\sin\theta} + \frac{2-3i\sin\theta}{1+2i\sin\theta} = 0$$

$$(2+3i\sin\theta)(1+2i\sin\theta) + (2-3i\sin\theta)(1-2i\sin\theta) = 0$$

$$2 + \cancel{4i\sin\theta} + \cancel{3i\sin\theta} + 6\sin^2\theta +$$

$$2 - \cancel{4i\sin\theta} - \cancel{3i\sin\theta} - 6\sin^2\theta = 0$$

$$4 - 12\sin^2\theta = 0$$

$$4(1-3\sin^2\theta) = 0$$

$$3\sin^2\theta = 1$$

$$\sin\theta = \pm\frac{1}{\sqrt{3}}$$

Value of θ for which z is purely imaginary $\theta = \sin^{-1}\dfrac{1}{\sqrt{3}}$

OR

2016 Mains

A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\frac{1}{\sqrt{3}}$

z is purely imaginary (given)

$\Rightarrow \quad Re(z) = 0$

$$z = \frac{2+3i\sin\theta}{1-2i\sin\theta}$$

$$\frac{2+3i\sin\theta}{1-2i\sin\theta} \times \frac{1+2i\sin\theta}{1+2i\sin\theta}$$

$$\frac{2+4i\sin\theta+3i\sin\theta-6\sin^2\theta}{1+4\sin^2\theta}$$

$$\frac{2-6\sin^2\theta}{1+4\sin^2\theta} + i\,\frac{7\sin\theta}{1+4\sin^2\theta}.$$

$$Re(z) = 0$$

$$\frac{2-6\sin^2\theta}{1+4\sin^2\theta} = 0$$

$$2-6\sin^2\theta = 0$$

$\sin\theta = \pm\frac{1}{\sqrt{3}}$ $\quad \theta = \sin^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)$

$\theta = \pm\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d)

2017 offline

Let ω be a complex number such that $2\omega + 1 = z$ where $z = \sqrt{-3}$

If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$ then k is equal

(a) $-z$ (b) z (c) -1 (d) 1

$$2\omega + 1 = z$$

$$2\omega = z - 1$$

$$\omega = \frac{1}{2}(z-1)$$

$$= \frac{1}{2}(\sqrt{-3} - 1) \quad [z = \sqrt{-3} \text{ given}]$$

$$= \frac{-1+\sqrt{-3}}{2}$$

$$\omega = \frac{-1+\sqrt{3}i}{2}$$

Since ω is cube root of unity.

$$\omega^2 = \frac{-1-\sqrt{3}i}{2} \quad \& \quad \omega^{3n} = 1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{bmatrix} = 3k$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} = 3k$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\begin{bmatrix} 3 & 1+\omega+\omega^2 & 1+\omega^2+\omega \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} = 3k$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} = 3k$$

$$3(\omega^2 - \omega^4) = 3k$$

$$\omega^2 - \omega^4 = k$$

$$\omega^2 - \omega^3.\omega = k$$

$$\omega^2 - \omega = k$$

$$\left(\frac{-1-\sqrt{3}i}{2}\right) - \left(\frac{-1+\sqrt{3}i}{2}\right) = k$$

$$-\sqrt{3}i = k$$

$$-2 = k \quad (a)$$