Maths

Question No 1 to 4 carry 1 mark each.

1. If A is a square matrix of order 3 with $|A| = 4$ Then what is the value of $|-2A|$

$$|-2A| = (-2)^3 |A| = -8(4) = -32$$

2. If $y = \sin^{-1}x + \cos^{-1}x$

$$y = \frac{\pi}{2} \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$\frac{dy}{dx} = 0.$$

3. Write order and degree

$$\left(\frac{d^4y}{dx^4}\right)^2 = \left[x + \left(\frac{dy}{dx}\right)^2\right]^3$$

Order $= 4$

degree $= 2$

4. If line has direction ratios $-18, 12, -4$ Then what are direction cosines

Direction cosines are.

$$\frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}},$$

$$\frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$$

$$\frac{-18}{\sqrt{324+144+16}}, \frac{12}{\sqrt{324+144+16}}, \frac{-4}{\sqrt{324+144+16}}$$

$$-\frac{\cancel{18}\,9}{\cancel{22}\,11}, \frac{\cancel{12}\,6}{\cancel{22}\,11}, -\frac{\cancel{4}\,2}{\cancel{22}\,11}$$

$$-\frac{9}{11}, \frac{6}{11}, -\frac{2}{11}.$$

or.

(4) Find the cartesian equation of a line which passes through the point $(-2, 4, 5)$ and parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{x+8}{6}$

$$\frac{x+3}{3} = \frac{4-y}{5} = \frac{x+8}{6}$$

$$\frac{x+3}{3} = \frac{-(y-4)}{5} = \frac{x+8}{6}$$

$$\frac{(x-(-3))}{3} = \frac{y-4}{-5} = \frac{x-(-8)}{6} \quad \text{I}$$

Required equation of line which passes through $(-2, 4, 5)$ and ~~which~~ which is parallel to the line I is

$$\frac{x-(-2)}{3} = \frac{y-4}{-5} = \frac{z-5}{6}$$

$$\frac{x+2}{3} = \frac{4-y}{5} = \frac{z-5}{6}$$

SECTION B

Questions number 5 to 12 carry 2 marks each

5. If $*$ is defined on the set R of all real numbers by $*$: $a * b = \sqrt{a^2 + b^2}$
Find the identity element if it exist in R w.r.t $*$

Let e be identity element for the binary operation in R

$$a * e = a = e * a$$

$$a * e = a$$

$$\sqrt{a^2 + e^2} = a$$

Squaring both side we get.

$$a^2 + e^2 = a^2$$

$$e^2 = 0$$

$$e = 0$$

$$e * a = a$$

$$\sqrt{e^2 + a^2} = a$$

Squaring both side

$$e^2 + a^2 = a^2$$

$$e^2 = 0$$

$$e = 0.$$

$\Rightarrow$ 0 is the identity element in R

6. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ $\quad kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$

Find the value of k, a, b.

$$KA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$k \begin{bmatrix} 4 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$\begin{bmatrix} 4k & 2k \\ 3k & -4k \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$-4k = 24$$

$$k = \frac{24}{-4} = -6$$

$$2k = 3a$$

$$2 \times -6 = 3a$$

$$-12 = 3a$$

$$-\frac{\cancel{12}^{4}}{\cancel{3}} = a$$

$$a = -4$$

$$3k = 2b$$

$$3 \times -6 = 2b$$

$$-18 = 2b$$

$$-9 = b$$

So $K = -6 \quad a = -4 \quad b = -9$

7 $\displaystyle\int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x}}\, dx$

$$\int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}}\, dx$$

$$\int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2}}\, dx$$

$$\int \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$$

Put: $\sin x + \cos x = t$

$(\cos x - \sin x)\, dx = dt$

$-(\sin x - \cos x)\, dx = dt$

$$\int -\frac{1}{t}\, dt$$

$- \log|t| + c$

$- \log|\sin x + \cos x| + c$

(8)

$$\int \frac{\sin(x-a)}{\sin(x+a)}\,dx$$

$$\int \frac{\sin(x-a+a-a)}{\sin(x+a)}\,dx$$

$$\int \frac{\sin((x+a)-2a)}{\sin(x+a)}\,dx$$

$$\int \frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)}\,dx$$

$$\int \left(\frac{\cancel{\sin(x+a)}\cos 2a}{\cancel{\sin(x+a)}} - \frac{\cos(x+a)\sin 2a}{\sin(x+a)}\right)dx$$

$$\int \cos 2a\,dx - \int \cot(x+a)\,\cancel{dx}\,\sin 2a\,dx.$$

$$\cos 2a \int 1\,dx - \cancel{\sin 2a}\int \cot(x+a)\,dx$$

$$\cos 2a\,(x) - \sin 2a \log|\sin(x+a)| + c$$

$$x\cos 2a - \sin 2a \log|\sin(x+a)| + c$$

or

8'

$$\int (\log x)^2 \, dx$$

$$\int \underset{\text{II}}{1} \, \underset{\text{I}}{(\log x)^2} \, dx.$$

$$(\log x)^2 \cdot x - \int 2 \log x \times \frac{1}{\cancel{x}} \cancel{x} \, dx.$$

$$x(\log x)^2 - \int 2 \log x \, dx$$

$$x(\log x)^2 - 2 \int \underset{\text{II}}{1} \, \underset{\text{I}}{\log x} \, dx$$

$$x(\log x)^2 - 2\left[(\log x) \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx\right]$$

$$x(\log x)^2 - 2\left[x \log x - x\right] + C$$

$$x(\log x)^2 - 2(x \log x - x) + C$$

9. Form the differential equation representing the family of curves $y^2 = m(a^2 - x^2)$ by eliminating the arbitrary constants m and a

$$y^2 = m(a^2 - x^2)$$

$$y^2 = ma^2 - mx^2$$

$$\cancel{2}y \frac{dy}{dx} = -\cancel{2}mx.$$

$$y \frac{dy}{dx} = -mx. \quad \text{I}$$

$$y \frac{d^2y}{dx^2} + 1 \cdot \frac{dy}{dx} = -m$$

Put this value in I

$$y \frac{dy}{dx} = x\left(y \frac{d^2y}{dx^2} + \frac{dy}{dx}\right)$$

10 Find a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$

$$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$$

$$\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$$

Sol:- $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$\hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21)$$

$$= 19\hat{j} + 19\hat{k} = 19\hat{j} + 19\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{19^2 + 19^2}$$

$$= \sqrt{2(19)^2} = 19\sqrt{2}$$

unit vector perpendicular both the vectors

$$\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$$

$$\frac{19\hat{j} + 19\hat{k}}{19\sqrt{2}}$$

$$\frac{19}{19\sqrt{2}}\hat{j} + \frac{19}{19\sqrt{2}}\hat{k}$$

$$\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

10 or

Show that the vectors

$\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$

$\hat{i} - 3\hat{j} + 5\hat{k}$ are co-planar

Prove $\begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$

$1(15 - 12) - (-2)(-10 + 4) + 3(6 - 3) = 0$

$3 + 2(-6) + 3(3) = 0$

$3 - 12 + 9 = 0$

$0 = 0$

which is true

so the vectors are co-planar.

11. Mother, Father and son line up at random for a family photograph. If A and B are two events given by A is son on one end and B father in the middle. Find $P(B|A)$

Sol. $S = \{MFS, MSF, FMS, FSM, SFM, SMF\}$

~~A = Son Son on~~

A = Son on one end

$A = \{MFS, FMS, SFM, SMF\}$

B = Father in the middle

$B = \{MFS, SFM\}$

$A \cap B = \{MFS, SFM\}$

$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$

$P(A) = \frac{4}{6} = \frac{2}{3}$

$P(B) = \frac{2}{6} = \frac{1}{3}$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

$= \frac{1/3}{2/3} = \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{2} = \frac{1}{2}$

12) Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X=x_1) = 3P(X=x_2) = P(X=x_3) = 5P(X=x_4)$

Find the probability distribution of X

$$2P(X=x_1) = 3P(X=x_2) = P(X=x_3) = 5P(X=x_4) = k$$

$$2P(X=x_1) = k$$

$$P(X=x_1) = \frac{k}{2}$$

$$3P(X=x_2) = k$$

$$P(X=x_2) = \frac{k}{3}$$

$$P(X=x_3) = k$$

$$5P(X=x_4) = k$$

$$P(X=x_4) = \frac{k}{5}$$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{15k + 10k + 30k + 6k}{30} = 1$$

$$61k = 30$$

$$k = \frac{30}{61}$$

X	1	2	3	4
$P(X)$	$\frac{1}{2} \times \frac{\cancel{30}^{15}}{61}$ $15/61$	$\frac{1}{3} \times \frac{\cancel{30}^{10}}{61}$ $10/61$	$\frac{30}{61}$ $30/61$	$\frac{1}{5} \times \frac{\cancel{30}^{6}}{61}$ $6/61$

or

12. A coin is tossed 5 times Find the probability of getting

I at least 4 heads

II at most 4 heads.

The repeated tosses of a coin are Bernoulli trials

Let X denote the number of heads

$n = 5 \quad p = \frac{1}{2} \quad q = \frac{1}{2}$

$P(\text{at least } 4 \text{ head}) = P(X=4) + P(X=5)$

$= {}^5C_4 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 + {}^5C_5 \left(\frac{1}{2}\right)^5$

$= 5\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5$

$= \left(\frac{1}{2}\right)^5 (6)$

$= \frac{1}{\cancel{32}_{16}} \times \cancel{6}^{3} = \frac{3}{16}$

$P(\text{at most } 4 \text{ head}) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

${}^5C_0 \left(\frac{1}{2}\right)^5 + {}^5C_1 \left(\frac{1}{2}\right)^5 + {}^5C_2 \left(\frac{1}{2}\right)^5 + {}^5C_3 \left(\frac{1}{2}\right)^5 + {}^5C_4 \left(\frac{1}{2}\right)^5$

$\left(\frac{1}{2}\right)^5 [1 + 5 + 10 + 10 + 5]$

$\frac{1}{32} \times 31 = \frac{31}{32}$

SECTION C

Questions number 13 to carry 4 marks each.

13. Show that the relation R in the set Z of all integers given by

$R = \{(a, b) : 2 \text{ divides } (a-b)\}$

is an equivalence relation.

Sol. Reflexive :-

2 divides $(a-a)$ for all $a \in Z$

So it is reflexive

Symmetric :-

If $(a, b) \in R \Rightarrow$ 2 divides $a-b$

$\Rightarrow$ 2 divides $b-a$

Hence $(b, a) \in R$

So it is symmetric

Transitive

If $(a, b) \in R \Rightarrow$ 2 divides $(a-b)$ $\quad a-b = 2q$

If $(b, c) \in R \Rightarrow$ 2 divides $(b-c)$

$b-c = 2q$.

$$\begin{aligned} a-c &= a+b+b-c \\ &= (a-b)+(b-c) \\ &= 2q + 2q \\ &= 4q \end{aligned}$$

$a-c = 4q \Rightarrow a-c$ is even

so $a-c$ is divisible by 2

$(a, c) \in R \Rightarrow R$ is transitive

14 Q- $\tan^{-1}(x) - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Find the value of x and hence find $\sec^{-1}\left(\frac{2}{x}\right)$

$$\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\tan\frac{\pi}{6}\right)$$

$$\tan^{-1} x - \cot^{-1} x = \frac{\pi}{6} \qquad \mathrm{I}$$

we know $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \qquad \mathrm{II}$

Adding I and II

$$2\tan^{-1} x = \frac{\pi}{6} + \frac{\pi}{2}$$

$$2\tan^{-1} x = \frac{2\cancel{4}\pi}{\cancel{6}3}$$

$$2\tan^{-1} x = \frac{2\pi}{3}$$

$$\tan^{-1} x = \frac{\pi}{3}$$

$$x = \sqrt{3}$$

Now $\sec^{-1}\left(\frac{2}{x}\right)$

$$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$\frac{\pi}{6}$$

15. Using properties of determinant.

Prove
$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

$$\Delta = \begin{vmatrix} 2b+2c & 2a+2c & 2a+2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$ Applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$ Take 2 comma from R_1

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$ Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$.

$$= 2\begin{vmatrix} 0 & c & b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$ Apply $R_1 \Rightarrow R_1 + R_2 + R_3$

$$2\left[(0) - c(-ab) + b(ac)\right]$$

$$2\left[abc + abc\right]$$

$$2(2abc)$$

$$4abc$$

16. If $\sin y = x \sin(a+y)$

Prove $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$

$$\sin y = x \sin(a+y)$$

$$\cos y \frac{dy}{dx} = x \cos(a+y)\frac{dy}{dx} + \sin(a+y)$$

$$\cancel{\frac{dy}{dx} = \frac{x\cos(a+y) + \sin(a+y)}{\cos y}}$$

$$\cos y \frac{dy}{dx} - x\cos(a+y)\frac{dy}{dx} = \sin(a+y)$$

$$\frac{dy}{dx}\left(\cos y - x\cos(a+y)\right) = \sin(a+y)$$

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)}$$

$$= \frac{\sin(a+y)}{\cos y - \frac{\sin y}{\sin(a+y)}\cos(a+y)}$$

$$= \frac{\sin(a+y) \times \sin(a+y)}{\cos y \sin(a+y) - \sin y \cos(a+y)}$$

$$= \frac{\sin^2(a+y)}{\sin(a+\cancel{y}-\cancel{y})} = \frac{\sin^2(a+y)}{\sin a}.$$

16. Q. $(\sin x)^y = x+y$ find $\frac{dy}{dx}$.

$$(\sin x)^y = x+y$$

Taking log on both side

$$\log(\sin x)^y = \log(x+y)$$

$$y\log(\sin x) = \log(x+y)$$

$$\frac{dy}{dx}\log(\sin x) + y\frac{1}{\sin x}\cos x = \frac{1}{x+y}\frac{d}{dx}(x+y)$$

$$\frac{dy}{dx}\log(\sin x) + y\cot x = \frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\log\sin x + y\cot x = \frac{1}{x+y} + \frac{1}{x+y}\frac{dy}{dx}.$$

$$\frac{dy}{dx}\log\sin x - \frac{1}{x+y}\frac{dy}{dx} = \frac{1}{x+y} - y\cot x$$

$$\frac{dy}{dx}\left(\log\sin x - \frac{1}{x+y}\right) = \frac{1}{x+y} - y\cot x$$

$$\frac{dy}{dx} = \frac{\frac{1}{x+y} - y\cot x}{\log\sin x - \frac{1}{x+y}}$$

$$= \frac{1 - y(x+y)\cot x}{(x+y)\log(\sin x) - 1}$$

17. If $y = (\sec^{-1}x)^2 \quad x > 0$

Show that: $x^2(x^2-1)\dfrac{d^2y}{dx^2} + (2x^3 - x)\dfrac{dy}{dx} - 2 = 0$

$$y = (\sec^{-1}x)^2$$

$$\frac{dy}{dx} = 2\sec^{-1}x \, \frac{d}{dx}\sec^{-1}x.$$

$$= 2\sec^{-1}x \, \frac{1}{x\sqrt{x^2-1}}$$

$$x\sqrt{x^2-1}\,\frac{dy}{dx} = 2\sec^{-1}x.$$

Differentiate

$$\left(x\sqrt{x^2-1}\right)\frac{d^2y}{dx^2} + \frac{d}{dx}\left(x\sqrt{x^2-1}\right)\frac{dy}{dx} = \frac{2}{x\sqrt{x^2-1}}$$

$$\left(x\sqrt{x^2-1}\right)\frac{d^2y}{dx^2} + \left(\sqrt{x^2-1} + x\,\frac{1}{\cancel{2}\sqrt{x^2-1}}\,\cancel{2}x\right)\frac{dy}{dx} = \frac{2}{x\sqrt{x^2-1}}.$$

$$x^2(x^2-1)\frac{d^2y}{dx^2} + \left(x(x^2-1) + \frac{x\cdot x^2\cancel{\sqrt{x^2-1}}}{\cancel{2}\cancel{(x^2-1)}}\right)\frac{dy}{dx} = 2.$$

$$x^2(x^2-1)\frac{d^2y}{dx^2} + \left(x^3 - x + x^3\right)\frac{dy}{dx} = 2$$

$$x^2(x^2-1)\frac{d^2y}{dx^2} + \left(2x^3 - x\right)\frac{dy}{dx} - 2 = 0$$